Ejercicios examen integrales

1 Integrales

a) $\int \frac{x-3}{x^3-x^2}\,dx$

$$\frac{x-3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)}$$

$A=2 \quad B=3 \quad C=-2$

$$\int \frac{x-3}{x^3-x^2}\,dx = \int \frac{A}{x}dx + \int \frac{B}{x^2}dx + \int \frac{C}{x-1}dx =$$

$$= \int \frac{2}{x}\,dx + \int \frac{3}{x^2}\,dx + \int \frac{-2}{x-1}\,dx = 2\ln|x| - \frac{3}{x} - 2\ln|x+1| + C$$

b) $$\int x \cdot 7^{x^2} dx = \frac{1}{2\ln 7}\int 2x \cdot \ln 7 \cdot 7^{x^2} dx = \frac{7^{x^2}}{2\ln 7} + C$$

c) $$\int \frac{\ln x + 1}{x(\ln^2 x - \ln x)}\,dx =$$

$\ln x = t$

$\frac{1}{x}dx = dt \Rightarrow dt = \frac{dx}{x}$

$$= \int \frac{t+1}{t^2-t}\,dt = \int \frac{A}{t}\,dt + \int \frac{B}{t-1}\,dt.$$

$$\frac{t+1}{t^2-t} = \frac{A}{t} + \frac{B}{t-1} = \frac{A(t-1)+Bt}{t(t-1)}$$

$\begin{cases} A=-1 \\ B=2 \end{cases}$

$$I = \int \frac{-1}{t}dt + \int \frac{2}{t-1}\,dt = -\ln|t| + 2\ln|t-1| =$$

$$= -\ln|\ln x| + 2\ln|\ln x - 1| + C$$

d) $\int (2x+1)\cdot e^{-x}\,dx \rightarrow$ Por partes.

$u = 2x+1 \Rightarrow du = 2\,dx$

$dv = e^{-x} \Rightarrow v = -e^{-x}$

$I = -(2x+1)\cdot e^{-x} - \int -e^{-x}\cdot 2\,dx = -(2x+1)e^{-x} + 2\int e^{-x}\,dx =$

$-(2x+1)e^{-x} - 2e^{-x} + C.$

e) $\int \frac{dx}{(x+5)\sqrt{x+1}} =$ $\quad \sqrt{x+1} = t \quad x+1 = t^2 \quad dx = 2t\,dt. \quad x = t^2-1$

$= \int \frac{2\cancel{t}\,dt}{(t^2-1+5)\cancel{t}} = \int \frac{2}{t^2+4}\,dt = 2\cdot\frac{1}{2}\operatorname{arctg}\left(\frac{t}{2}\right) = \operatorname{arctg}\frac{t}{2} =$

$= \operatorname{arctg}\frac{\sqrt{x+1}}{2} + C$

2

a) Calcula el valor de a sabiendo que $\underline{a>0}$

$$\int_0^a \frac{-4x}{(1+x^2)^2}\,dx = -1$$

$$\int \frac{-4x}{(1+x^2)^2}\,dx = \int \frac{-4\cancel{x}\,dt}{t^2\cdot 2\cancel{x}} = -2\int \frac{dt}{t^2} = -2\left(-\frac{1}{t}\right) = \frac{2}{t} = \frac{2}{1+x^2}$$

$1+x^2 = t.$

$2x\,dx = dt.$

$dx = \frac{dt}{2x}$

$$\int_0^a \frac{-4x}{(1+x^2)^2}\,dx = -1$$

$$\left[\frac{2}{1+x^2}\right]_0^a = \frac{2}{1+a^2} - 2 = -1.$$

$\frac{2}{1+a^2} = 1 \quad a^2+1 = 2 \quad a^2 = 1 \quad a = \pm 1$ si $a>0$ $\boxed{a=1}$

b) Razona y halla los máx y mínimos relativos de

$$F(x) = \int_1^{x^2} (t^2 - 1)\,dt \qquad \begin{array}{l} x^2 = t \\ 2x\,dx = dt. \end{array}$$

Por el teorema del fundamental del cálculo integral

$$\int_a^x f(t)\,dt \longrightarrow \boxed{f(x) = F'(x)}$$

$$F'(x) = (x^4 - 1)\,2x \qquad F'(x) = 2x^5 - 2x$$

Condición de máx o mínimo

$$F'(x) = 0 \qquad (x^4 - 1)\,2x = 0 \quad \begin{cases} x = 0 \\ x = 1 \\ x = -1 \end{cases}$$

$$F''(x) = 10x^4 - 2$$

$\underline{x=0}$ $\quad F''(0) = -2 < 0$ $\qquad$ Máx relativo.

$\underline{x=1}$ $\quad F''(1) = 8 > 0$ $\qquad$ Mínimo rel.

$\underline{x=-1}$ $\quad F''(-1) = 8 > 0$ $\qquad$ Mínimo relat.

c) $f(x) = \frac{x}{2} + 2$. Calcula el valor medio de la func. en el intervalo $[0, 4]$

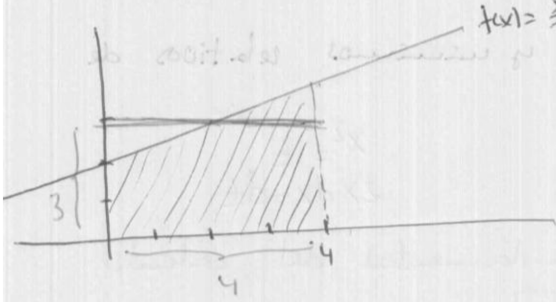
T. V. M. $\int_a^b f(x)\,dx = f(c)(b-a)$

$$\int_0^4 \left(\frac{x}{2} + 2\right) dx = \left[\frac{x^2}{4} + 2x\right]_0^4 = \frac{16}{4} + 8 = 12.$$

$$12 = f(c) \cdot (4 - 0) \implies f(c) = \frac{12}{4} = 3 \quad \boxed{f(c) = 3}$$

$$\frac{1}{2}c + 2 = 3$$

$$\frac{1}{2}c = 1 \quad \boxed{c = 2}$$



El área del recinto que se determina entre f(x) y el eje X en el intervalo $[0, 4]$ es un trapecio.

Es equivalente al área de un rectángulo de base $\underline{4}$ y altura $\underline{\underline{3}}$

3) $f(x) = x^3 - x + 1$

Área de la región limitada por f(x) y la recta tangente a la gráfica en $x = 1$

① Recta tg en $x = 1$ $\qquad f(1) = 1$

$$y - f(1) = f'(1)(x - 1)$$

$f'(x) = 3x^2 - 1 \quad \rightarrow f'(1) = 2 \qquad \boxed{f'(1) = m}$

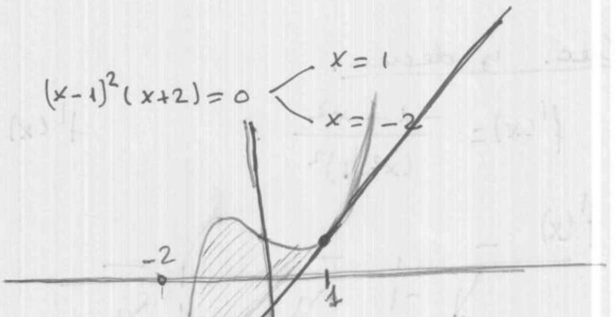
$y - 1 = 2(x - 1) = 2x - 2 \qquad \boxed{y = 2x - 1}$

② Ptos intersección entre f(x) y recta tg $y = 2x - 1$

$x^3 - x + 1 = 2x - 1$

$x^3 - 3x + 2 = 0 \qquad (x-1)^2(x+2) = 0 \quad \begin{cases} x = 1 \\ x = -2 \end{cases}$

	1	0	-3	2
1		1	1	-2
	1	1	-2	$\boxed{0}$
1		1	2	
	1	2	$\boxed{0}$	



$$A_T = \int_{-2}^{1} \left((x^3 - x + 1) - (2x - 1) \right) dx = \int_{-2}^{1} x^3 - 3x + 2 \, dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^{1} =$$

$$= \left[\frac{1}{4} - \frac{3}{2} + 2 \right] - (4 - 6 - 4) = \frac{3}{4} + 6 = \underline{\underline{\frac{27}{4}\ u^2}}$$

4) $f(x) = \dfrac{x}{x^2-1}$ $\qquad$ Dom $f(x) = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Ptos corte $\quad P(0,0)$

a) gráfica.

$f(x)$ simétrica respecto $O(0,0)$

↳ Impar

A. vert. $\begin{cases} x = 1 \\ x = -1 \end{cases}$

$\lim\limits_{x \to 1^+} \dfrac{x}{x^2-1} = +\infty$ $\qquad$ $\lim\limits_{x \to -1^+} \dfrac{x}{x^2-1} = +\infty$

$\lim\limits_{x \to 1^-} \dfrac{x}{x^2-1} = -\infty$ $\qquad$ $\lim\limits_{x \to -1^-} \dfrac{x}{x^2-1} = -\infty$

A. H.

$\lim\limits_{x \to +\infty} \dfrac{x}{x^2-1} = 0$ $\qquad$ $\boxed{y = 0}$

Crec. y decrec.

$f'(x) = \dfrac{-1-x^2}{(x^2-1)^2}$ $\qquad$ $f'(x) = 0 \quad -1 - x^2 = 0$

$x^2 + 1 = 0$

↓

No se anula.

$f'(x)$: $\quad - \quad | -1 | \quad - \quad | 1 | \quad -$

$\quad\searrow \qquad\quad \searrow \qquad\quad \searrow$

No tiene máx ni mínimos

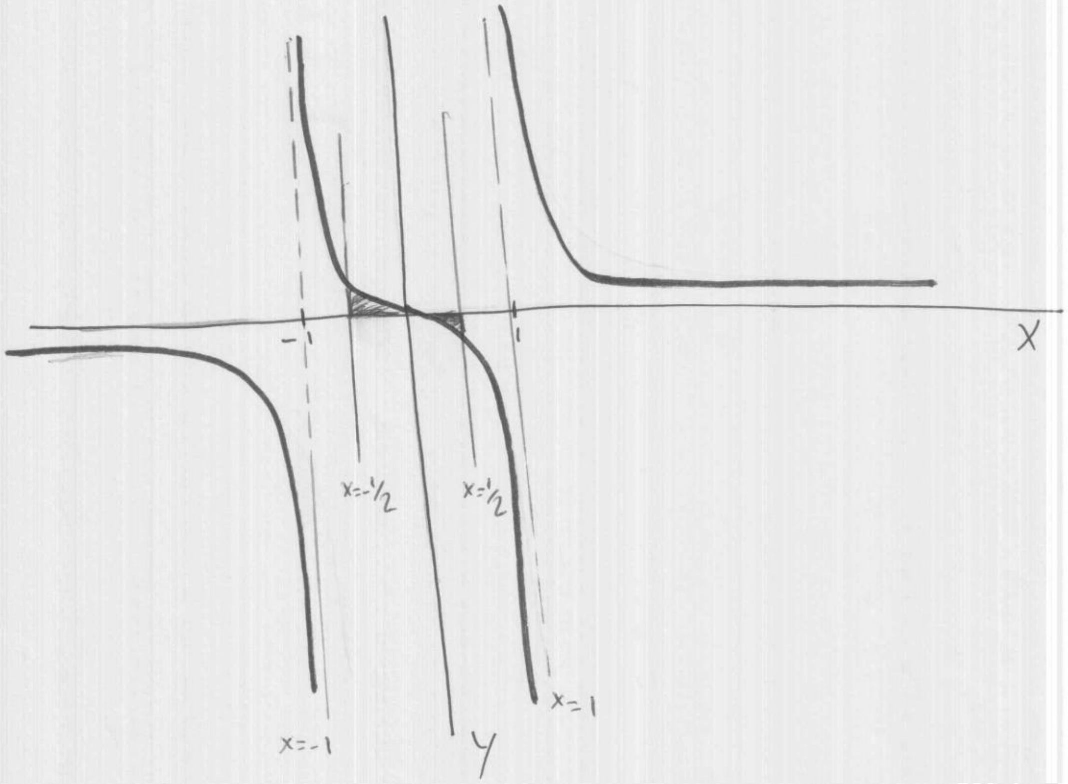
Siempre es decreciente.

Ptos Inflexión

$f''(x) = \dfrac{2x^3+6x}{(x^2-1)^3}$ $\qquad$ $f''(x) = 0 \quad (2x^2+6)x = 0 \Rightarrow \boxed{x = 0}$

P.I $(0,0)$

b) Area $y = f(x)$ Eje Ox rectas $x = -\frac{1}{2}$ $x = \frac{1}{2}$



$A_T = A_1 + A_2 = 2A_1$ $A_1 = A_2$.

$$A_T = 2\int_{-1/2}^{0} \frac{x}{x^2-1}\,dx = \int_{-1/2}^{0} \frac{2x}{x^2-1}\,dx = \Big[\ln|x^2-1|\Big]_{-1/2}^{0}$$

$$= 0 - \ln\frac{3}{4} = \ln\frac{4}{3} = \underline{\underline{0{,}28\ u^2}}$$

4 de 4

$$f(x) = \frac{x}{e^x - 1}$$

Ptos corte → No tu.

Dom $f(x) = \mathbb{R} - \{0\}$

A.V → NO

$$\lim_{x \to 0} \frac{x}{e^x - 1} = \frac{0}{0} \overset{L.H}{=} \lim_{x \to 0} \frac{1}{e^x} = 1$$ $\boxed{x=0}$ No es A.V

A.H →

$$\lim_{x \to +\infty} \frac{x}{e^x - 1} = \frac{\infty}{\infty} \overset{LH}{=} \lim_{x \to \infty} \frac{1}{e^x} = 0$$ $\boxed{y=0 \quad x \to +\infty}$ A.H.

$$\lim_{x \to -\infty} \frac{x}{e^x - 1} = +\infty$$ → Rama infinita. $\boxed{x \to -\infty \quad f(x) \to +\infty}$

A. Oblicuas

Puede haber $x \to -\infty$.

$$m = \lim_{x \to -\infty} \frac{x}{x(e^x - 1)} = \lim_{x \to -\infty} \frac{1}{e^x - 1} = \frac{1}{-1} = -1$$ $\boxed{m=-1}$

$$n = \lim_{x \to -\infty} \left(\frac{x}{e^x - 1} + x \right) = \lim_{x \to -\infty} \left(\frac{x + x.e^x - x}{e^x - 1} \right) = \lim_{x \to -\infty} \left(\frac{x e^x}{e^x - 1} \right) = -\infty \cdot 0$$

$$= \lim_{x \to -\infty} \frac{x}{\frac{1}{e^x}(e^x - 1)} = \lim_{x \to -\infty} \frac{x}{\frac{e^x}{e^x} - \frac{1}{e^x}} = \lim_{x \to -\infty} \frac{x}{1 - \frac{1}{e^x}} = \left(\frac{-\infty}{-\infty} \right) \overset{L.H}{=}$$

$$= \lim_{x \to -\infty} \frac{1}{e^{-x}} = 0$$

$\boxed{y = -x}$ A. Oblicua

Crec. y decrec.

$$f'(x) = \frac{e^x - 1 - x e^x}{(e^x - 1)^2} = \frac{(1-x) e^x - 1}{(e^x - 1)^2}$$

$f'(x) \neq 0$ → No tu. ptos singulares

$$f'(x) \quad \begin{array}{ccc} - & | & - \\ \searrow & 0 & \searrow \end{array}$$

- Decreciente.
- No tu. mín. ni máx

(4) $f(x) = \dfrac{x}{x^2-1}$ Dom $f(x) = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Ptos corte $P(0,0)$

$f(x)$ simétrica respecto $O(0,0)$

↳ Impar

a) gráfica.

A. vert. $x = 1$, $x = -1$

$\lim\limits_{x \to 1^+} \dfrac{\cancel{x}}{x^2-1} = +\infty$ $\lim\limits_{x \to -1^+} \dfrac{x}{x^2-1} = +\infty$

$\lim\limits_{x \to 1^-} \dfrac{x}{x^2-1} = -\infty$ $\lim\limits_{x \to -1^-} \dfrac{x}{x^2-1} = -\infty$

A. H.

$\lim\limits_{x \to +\infty} \dfrac{x}{x^2-1} = 0$ $\boxed{y = 0}$

Crec. y decrec.

$f'(x) = \dfrac{-1-x^2}{(x^2-1)^2}$ $f'(x) = 0$ $-1-x^2 = 0$

$x^2+1 = 0$

↓

No se anula.

No tiene máx ni mínimos

		-1		1	
$f'(x)$	$-$		$-$		$-$
	↘		↘		↘

Siempre es decreciente.

Ptos Inflexión

$f''(x) = \dfrac{2x^3+6x}{(x^2-1)^3}$ $f''(x) = 0$ $(2x^2+6)x = 0 \Rightarrow \boxed{x = 0}$

P.I $(0,0)$.

Ejercicios de Representación de funciones