Ejercicios de números complejos 1º bachilllerato

7.28

a) $\frac{(1+i)^2}{4+i} = \frac{1+2i+i^2}{4+i} = \frac{2i}{4+i} = \frac{2i(4-i)}{(4+i)(4-i)} = \frac{2+8i}{16+1} = \frac{2}{17} + \frac{8}{17}i$

b) $\frac{1}{i^9} + 5i^9 = \frac{1}{i} + 5i = \frac{i}{i^2} + 5i = -i + 5i = 4i$

c) $(i^5 + i^{-12})^3 = \left(i + \frac{1}{i^{12}}\right)^3 = (i+1)^3 = i^3 + 3i^2 + 3i + 1 = -i - 3 + 3i + 1 = -2 + 2i$

d) $\left(\frac{2i^5 + 3i^{17}}{1+i}\right)^2 = \left(\frac{2i+3i}{1+i}\right)^2 = \left(\frac{5i}{1+i}\right)^2 = \frac{25i^2}{(1+i)^2} = \frac{-25}{2i} = \frac{25}{2}i$

7.29

a) $(-i)^{361} = -i^{361} = -i^{4\cdot 90+1} = -i$

b) $i^{-346} = \frac{1}{i^{346}} = \frac{1}{i^{4\cdot 86+2}} = \frac{1}{i^2} = -1$

c) $(-i)^{-15} = \frac{1}{-i^{15}} = \frac{1}{-i^{3\cdot 4+3}} = \frac{1}{-i^3} = -i$

d) $\frac{1}{i^{33}} = \frac{1}{i^{4\cdot 8+1}} = \frac{1}{i} = \frac{i}{-1} = -i$

e) $(i^{3\,742\,359\,768})^4 = 1$

f) $\frac{-1}{(-i)^{11}} = \frac{-1}{-i^{4\cdot 2+3}} = \frac{1}{i^3} = i$

7.33

a) $(1+i)^5 = (\sqrt{2}_{45°})^5 = (4\sqrt{2})_{5\cdot 45°} = (4\sqrt{2})_{225°} = 4\sqrt{2}(\cos 225° + i \operatorname{sen} 225°) = 4\sqrt{2}\left(\frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -4 - 4i$

b) $(2 + 2\sqrt{3}i)^2 = 4 + 8\sqrt{3}i + 4\cdot i^2 \cdot 3 = 4 - 12 + 8\sqrt{3}i = -8 + 8\sqrt{3}i$

c) $(1+i)^{20} = (\sqrt{2}_{45°})^{20} = (\sqrt{2})^{20}_{20\cdot 45°} = 1024_{900°} = 1024_{180°} = -1024$

d) $(2 + 2\sqrt{3}i)^6 = (4_{60°})^6 = (4^6)_{6\cdot 60°} = 4096_{0°} = 4096$

7.34

a) $\sqrt[3]{-1} = \sqrt[3]{1_{180°}}$ $\begin{cases} 1_{60°} \\ 1_{180°} = -1 \\ 1_{300°} \end{cases}$

b) $\sqrt[4]{1+i} = \sqrt[4]{\sqrt{2}_{45°}}$ $\begin{cases} \sqrt[8]{2}_{45°/4} \\ \sqrt[8]{2}_{405°/4} \\ \sqrt[8]{2}_{765°/4} \\ \sqrt[8]{2}_{1125°/4} \end{cases}$

c) $\sqrt{-36} = \sqrt{36}\cdot\sqrt{-1} = \pm 6i$

d) $\sqrt[3]{-27} = \sqrt[3]{27_{180°}}$ $\begin{cases} 3_{60°} \\ 3_{180°} = -3 \\ 3_{300°} \end{cases}$

e) $\sqrt[6]{729i} = \sqrt[6]{729_{90°}}$ $\begin{cases} 3_{15°} \\ 3_{75°} \\ 3_{135°} \\ 3_{195°} \\ 3_{255°} \\ 2_{315°} \end{cases}$

f) $\sqrt[4]{16(\cos 180° + i \operatorname{sen} 180°)} = \sqrt[4]{16_{180°}}$ $\begin{cases} 2_{45°} \\ 2_{135°} \\ 2_{225°} \\ 2_{315°} \end{cases}$

7.35

$$(-1+i)^{30} = \left(\sqrt{2}_{135^\circ}\right)^{30} = \left(\sqrt{2}\right)^{30}_{30\cdot135^\circ} = 2^{15}\,(\cos 4050^\circ + i \operatorname{sen} 4050^\circ) = 2^{15}\,(\cos 90^\circ + i \operatorname{sen} 90^\circ) =$$
$$= 2^{15}\,(0 + i1) = 2^{15}\, i = 32\,768i$$

7.40

$$\frac{i^5 - i^3}{1+i} = \frac{i-(-i)}{1+i} = \frac{2i}{1+i} = \frac{2i(1-i)}{(1+i)(1-i)} = 1 + i = \sqrt{2}_{45^\circ} \rightarrow \sqrt[4]{\sqrt{2}_{45^\circ}} = \sqrt[8]{2}_{\frac{45^\circ+360^\circ k}{4}} = \begin{cases} \sqrt[8]{2}_{11^\circ15'} \\ \sqrt[8]{2}_{101^\circ15'} \\ \sqrt[8]{2}_{191^\circ15'} \\ \sqrt[8]{2}_{281^\circ15'} \end{cases}$$

7.41

a) $\dfrac{27\sqrt{3}}{2} + \dfrac{27}{2}i = 27_{30^\circ} \quad \Rightarrow \quad \sqrt[3]{27_{30^\circ+n\cdot360^\circ}} = 3_{\frac{30^\circ+360^\circ\cdot n}{3}} = \{3_{10^\circ}, 3_{130^\circ}, 3_{250^\circ}\}$

b) $4 + 4\sqrt{3}\,i = 8_{60^\circ} \quad \Rightarrow \quad \sqrt[3]{8_{30^\circ+360^\circ\cdot n}} = 2_{\frac{60^\circ+360^\circ\cdot n}{3}} = \{2_{20^\circ}; 2_{140^\circ}; 2_{260^\circ}\}$

c) $\sqrt{2} + \sqrt{2}\,i = 2_{45^\circ} \quad \Rightarrow \quad \sqrt[3]{2_{45^\circ+360^\circ\cdot n}} = \{\sqrt[3]{2}_{15^\circ}; \sqrt[3]{2}_{135^\circ}; \sqrt[3]{2}_{225^\circ}\}$

d) $5 - 5i = 5\sqrt{2}_{45^\circ} \quad \Rightarrow \quad \sqrt[3]{5\sqrt{2}_{45^\circ+360^\circ\cdot n}} = \sqrt[6]{50}_{\frac{45^\circ+360^\circ\cdot n}{3}} = \{\sqrt[6]{50}_{15^\circ}; \sqrt[6]{50}_{135^\circ}; \sqrt[6]{50}_{255^\circ}\}$

Ejercicios de vectores

4.47

a) $|\vec{u}| = \sqrt{(-\cos a)^2 + (\text{sen } a)^2} = \sqrt{\cos^2 a + \text{sen}^2 a} = 1.$

$\vec{v} = (\text{sen } a, \cos a)$, ya que $\vec{u} \cdot \vec{v} = 0$

$\vec{w} = (-\text{sen } a, -\cos a).$

4.48

$$\left.\begin{array}{l} \vec{u} \cdot \vec{v} = 0 \Rightarrow (x, y) \cdot (3, -4) = 3x - 4y = 0 \\ \vec{u} \cdot \vec{w} = 2 \Rightarrow (x, y)\ (2, -3) = 2x - 3y = 2 \end{array}\right\} \Rightarrow x = -8,\ y = -6.$$ Por tanto, el vector es $\vec{u} = (-8, -6)$.

4.49

a) $\vec{u} = (5, 2)$; $\quad \vec{v} = (-3, 3) \quad \vec{u} + \vec{v} = (2, 5)$

b) $|\vec{u}| = \sqrt{5^2 + 2^2} = \sqrt{29}$; $|\vec{v}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$; $|\vec{u} + \vec{v}| = \sqrt{2^2 + 5^2} = \sqrt{29}$

c) $\vec{u} \cdot \vec{v} = (5, 2) \cdot (-3, 3) = -15 + 6 = -9$

d) Proyección de $\vec{u}$ sobre $\vec{v} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \dfrac{-9}{3\sqrt{2}} = \dfrac{-3}{\sqrt{2}}$

e) $\cos(\widehat{\vec{u}, \vec{v}}) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|} = \dfrac{-9}{\sqrt{29} \cdot 3\sqrt{2}} = \dfrac{-3}{\sqrt{58}} \Rightarrow (\widehat{\vec{u}, \vec{v}}) = \text{arc cos}\left(\dfrac{-3}{\sqrt{58}}\right) = 113°\ 11'\ 54{,}93''$

f) Basta multiplicar el vector $\vec{u}$ por el inverso de su módulo: $\vec{w} = \dfrac{\vec{u}}{|\vec{u}|} = \left(\dfrac{5}{\sqrt{29}}, \dfrac{2}{\sqrt{29}}\right)$.

g) El vector $\vec{p} = (-2, 5)$ es ortogonal a $\vec{u}$, ya que $\vec{p} \cdot \vec{u} = 0$. Por tanto, $\left(\dfrac{-2}{\sqrt{29}}, \dfrac{5}{\sqrt{29}}\right)$ es unitario y ortogonal a $\vec{u}$.

4.50

a) $\vec{u} \cdot \vec{v} = (3, 4) \cdot (4, 3) = 12 + 12 = 24$

b) $|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ y $|\vec{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

c) $\cos(\widehat{\vec{u}, \vec{v}}) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|} = \dfrac{24}{25} \Rightarrow (\widehat{\vec{u}, \vec{v}}) = \text{arc cos}\left(\dfrac{24}{25}\right) = 16°\ 15'\ 36{,}74''$

d) Proyección de $\vec{u}$ sobre $\vec{v} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \dfrac{24}{5}$

e) El vector $-\dfrac{\vec{v}}{|\vec{v}|} = \left(-\dfrac{4}{5}, -\dfrac{3}{5}\right)$ es unitario y tiene la dirección de $\vec{v}$ y el sentido opuesto.

4.53

a) $\vec{u} = 4\vec{i} + \vec{j} \Rightarrow \vec{u} = (4, 1)$; $\vec{v} = -3\vec{i} + 3\vec{j} \Rightarrow \vec{v} = (-3, 3)$

b) $\vec{u} \cdot \vec{v} = (4, 1) \cdot (-3, 3) = -12 + 3 = -9$

c) $\cos(\widehat{\vec{u}, \vec{v}}) = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|} = \dfrac{-9}{\sqrt{4^2 + 1^2}\sqrt{3^2 + 3^2}} = \dfrac{-9}{\sqrt{17}\sqrt{18}} = \dfrac{-9}{\sqrt{306}} = \dfrac{-3}{\sqrt{34}}$, $(\widehat{\vec{u}, \vec{v}}) = \text{arc cos}\left(\dfrac{-3}{\sqrt{34}}\right) = 120°\ 57'\ 49{,}5''$

d) $\vec{u} + \vec{v} = (4, 1) + (-3, 3) = (1, 4)$

Un vector ortogonal al vector $\vec{u} + \vec{v} = (1, 4)$ puede ser $(-4, 1)$, y para que sea unitario basta con dividir por su módulo:

$\vec{w} = \left(-\dfrac{4}{\sqrt{17}}, \dfrac{1}{\sqrt{17}}\right)$